Lecture 4 — Basic GW Theory — SK

IUCAA Refresher Course 19–25 May 2026

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Quadrupole formula:

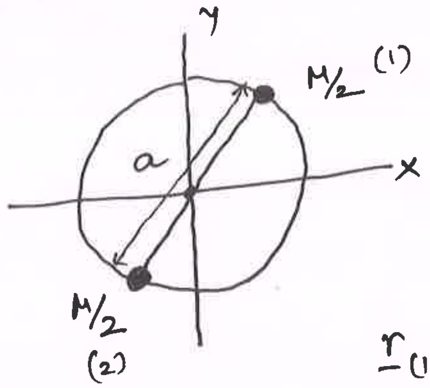
$$h_{\alpha\beta}^{TT} = \frac{2G}{c^4 r} \ddot{I}_{\alpha\beta}^{TT}$$

where $\quad I_{\alpha\beta} = \int T_{00}\, x'_\alpha x'_\beta \, d^3 r'$

$I_{\alpha\beta}^{TT}$, $\ddot{I}_{\alpha\beta}^{TT}$ is obtained by using the projection discussed earlier.

We will discuss other methods of deriving the quadrupole formula at the end of this lecture.

Let us now do an application to a simple system : an equal mass circular binary



$a = 2R$

Masses $M/2$ each.

Total mass M.

Coordinates :

$$\underline{r}_{(1)} = \left[R\cos\omega t_r ,\ R\sin\omega t_r ,\ 0 \right]$$

$$\underline{r}_{(2)} = \left[-R\cos\omega t_r ,\ -R\sin\omega t_r ,\ 0 \right]$$

Let us now find $I_{\alpha\beta}$. The ρ is given as a sum $\frac{M}{2}\delta^{(3)}(\underline{r}' - \underline{r}_1) + \frac{M}{2}\delta^{(3)}(\underline{r}' - \underline{r}_2)$.

$$I_{xx} = \frac{M}{2}R^2\cos^2\omega t_r + \frac{M}{2}R^2\cos^2\omega t_r = MR^2\cos^2\omega t_r$$

$$= \frac{MR^2}{2}(1 + \cos 2\omega t_r).$$

$$I_{yy} = \frac{M}{2}R^2\sin^2\omega t_r + \frac{M}{2}R^2\sin^2\omega t_r = \frac{MR^2}{2}(1 - \cos 2\omega t_r).$$

$$I_{xy} = \frac{M}{2} R^2 \sin\omega t_r \cos\omega t_r \times 2$$

$$= \frac{MR^2}{2} \sin 2\omega t_r .$$

(2)

$$\ddot{I}_{xx} = -\frac{MR^2}{2}(2\omega)^2 \cos 2\omega t_r = -2MR^2 \cos 2\omega t_r$$

$$\ddot{I}_{yy} = +\frac{MR^2}{2}(2\omega)^2 \cos 2\omega t_r . = 2MR^2 \cos 2\omega t_r$$

$$\ddot{I}_{xy} = -\frac{MR^2}{2}(2\omega)^2 \sin 2\omega t_r . = -2MR^2 \sin 2\omega t_r .$$

others are zero.

$$\therefore \ddot{I}_{\alpha\beta} = 2MR^2\omega^2 \begin{bmatrix} -\cos 2\omega t_r & -\sin 2\omega t_r & 0 \\ -\sin 2\omega t_r & \cos 2\omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix} = \ddot{I}^{TT}_{\alpha\beta}$$

This is already in the TT form. No further projection required.

$$\therefore h^{TT}_{\alpha\beta} = \frac{2G}{c^4 r} \cdot 2MR^2\omega^2 \begin{bmatrix} -\cos 2\omega t_r & -\sin 2\omega t_r & 0 \\ -\sin 2\omega t_r & \cos 2\omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\| h^{TT}_{\alpha\beta} \| . = \frac{4GMR^2\omega^2}{c^4 r}$$

Note : Frequency of GW is 2ω

$T = 1\,hr = 3600\,s$

$M = 2M_\odot$

Use $M/2 = 1 M_\odot$, $r = 1$ kpc. and make an estimate of $\| h^{TT}_{\alpha\beta} \|$.

<u>Kepler's 3rd Law</u>

$$\frac{G \frac{M}{2} \frac{M}{2}}{4R^2} = \frac{\frac{M}{2} \omega^2 R^2}{R} \qquad (3)$$

$$\Rightarrow \quad \frac{GM}{8R^2} = \omega^2 R$$

$$GM = \omega^2 8R^3 = \omega^2 a^3$$

$$\| h_{\alpha\beta}^{TT} \| = \frac{GM a^2 \omega^2}{c^4 r}$$

$$= \frac{GM a^3 \omega^2}{c^4 r \cdot a} = \frac{GM}{c^2 r} \cdot \frac{GM}{c^2 a}$$

$$\therefore \quad h_+^{TT} = -\left(\frac{GM}{c^2 r}\right)\left(\frac{GM}{c^2 a}\right) \cos 2\omega t_r$$

$$h_\times^{TT} = -\left(\frac{GM}{c^2 r}\right)\left(\frac{GM}{c^2 a}\right) \sin 2\omega t_r$$

One can also write :

$$\| h_{\alpha\beta}^{TT} \| = \frac{GM \left(\frac{GM}{\omega^2}\right)^{2/3} \omega^2}{c^4 r}$$

$$= \frac{G^{5/3} M^{5/3} \omega^{2/3}}{c^4 r}$$

Plug in $M = 2 \times 2 \times 10^{30}$ kg, $\omega = \frac{2\pi}{3600}$ s^{-1}

$c = 3 \times 10^8$ m/s, $r = 3.08 \times 10^{19}$ m $= 1$ kpc.

This gives $\| h_{\alpha\beta}^{TT} \| = \underline{\underline{1.285 \times 10^{-21}}}$

Energy of gravitational waves.

Principle of equivalence tells us that there is no way to define a 'local gravitational energy' because locally, gravity is not there. So, how does one define energy due to gravitational waves ?

The way out seems to define energy of GWs in an average sense — typically averaged over a region. The procedure is known as Brill-Hartle averaging and was used in the GW context by R. Isaacson in 1968 to define energy density or a T_{ij} due to gravitational waves.

To do this let us follow Brill-Hartle-Isaacson. Einstein tensor to order quadratic in h_{ij}

$$G_{ij} = G_{ij}^{(0)} + G_{ij}^{(1)}[h] + G_{ij}^{(2)}[h] + \cdots = 0$$

For a Minkowski background $G_{ij}^{(0)} = 0$

Let us write the metric

$$g_{ij} = \eta_{ij} + \epsilon h_{ij}^{(1)} + \epsilon^2 h_{ij}^{(2)} + \cdots$$

Then
$$0 = \epsilon G_{ij}^{(1)}[h^{(1)}] + \epsilon^2 G_{ij}^{(1)}[h^{(2)}] + \epsilon^2 G_{ij}^{(2)}[h^{(1)}] + \cdots$$

$$0 = \epsilon G_{ij}^{(1)}[h^{(1)}] + \epsilon^2 \left[G_{ij}^{(1)}[h^{(2)}] + G_{ij}^{(2)}[h^{(1)}] \right]$$

This must hold good at every order in ϵ.

This gives us $G^{(1)}_{ij}[h^{(1)}] = 0$

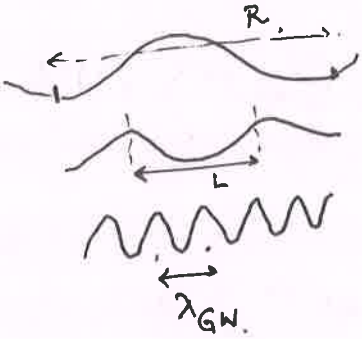
{Linearised gravity}

$$G^{(1)}_{ij}[h^{(2)}] = -G^{(2)}_{ij}[h^{(1)}]$$ and so on.

We write $$G^{(1)}_{ij}[h^{(2)}] = \frac{8\pi G}{c^4} T^{GW}_{ij}$$

or, $$T^{GW}_{ij} = -\frac{c^4}{8\pi G}\left\langle G^{(2)}_{ij}[h^{(1)}]\right\rangle_{avg.}$$

This is the effective T^{GW}_{ij} of gravitational waves after averaging.



$$\lambda_{GW} < L < R$$

λ_{GW} → wavelength of GW

L → avging region

R → curvature scale

One can work out the full $G^{(2)}_{ij}[h^{(1)}]$ and do the averaging. We will follow a simpler and easier route and proceed by extrapolation. Rigorous results are there in various articles referred to earlier.

Take $$ds^2 = -dt^2 + (1 + h_+(t,z))\,dx^2 + (1 - h_+(t,z))\,dy^2 + dz^2$$

One can check that $$R^{(2)}_{00} = h_+\ddot{h}_+ + \frac{\dot{h}_+^2}{2} = R^{(0)}_{33}$$

$R^{(2)}_{03} = -R^{(2)}_{00}$. There is a

and $R^{(2)}_{11} = R^{(2)}_{22} = 0$.

We have $R^{(2)} = 0$.

Now $$\left\langle R^{(2)}_{00}\right\rangle = \left\langle h_+\ddot{h}_+ + \frac{\dot{h}_+^2}{2}\right\rangle = -\left\langle \frac{\dot{h}_+^2}{2}\right\rangle.$$

$$= \left\langle G^{(2)}_{00}\right\rangle \quad \text{since } R^{(2)} = 0$$

Extrapolating from here one gets

$$T_{00}^{GW} = \frac{c^4}{32\pi G} \left\langle \partial_0 h^{ij\,TT} \partial_0 h_{ij}^{TT} \right\rangle$$

or, $$T_{kl}^{GW} = \frac{c^4}{32\pi G} \left\langle \partial_k h^{ij\,TT} \partial_l h_{ij}^{TT} \right\rangle$$

This is the Isaacson stress energy

It is gauge invariant (approx) not exact and conserved.

Let us now calculate $\frac{dE}{dt} = -L_{GW}$

$$\frac{dE}{dt} = -\frac{c^3}{32\pi G} \int \left\langle \dot{h}_{\alpha\beta}^{TT} \dot{h}^{\alpha\beta\,TT} \right\rangle r^2 d\Omega$$

or, $$\frac{dE}{dt} = -\frac{c^3}{32\pi G} \cdot \left(\frac{2G}{c^4 r}\right)^2 \int \left\langle \dddot{I}_{\alpha\beta}^{TT} \dddot{I}^{\alpha\beta\,TT} \right\rangle r^2 d\Omega$$

$$= -\frac{G}{8\pi c^5} \int \left\langle \dddot{I}_{\alpha\beta}^{TT} \dddot{I}^{\alpha\beta\,TT} \right\rangle d\Omega.$$

We will now have to do the integral over the sphere to get to the final expression.

$$I_{\alpha\beta}^{TT} = \left(P^\gamma_\alpha P^\delta_\beta - \frac{1}{2} P_{\alpha\beta} P^{\gamma\delta}\right) I_{\gamma\delta}$$

or, $$\dddot{I}_{\alpha\beta}^{TT} = \left(P^\gamma_\alpha P^\delta_\beta - \frac{1}{2} P_{\alpha\beta} P^{\gamma\delta}\right) \dddot{I}_{\gamma\delta}$$

We use these and work out the integral.

We have

$$\frac{dE}{dt} = -\frac{G}{8\pi c^5}\int \left(P^{\gamma}_{\alpha} P^{\delta}_{\beta} - \frac{1}{2} P_{\alpha\beta} P^{\gamma\delta}\right)\left(P^{\gamma'\alpha} P^{\delta'\beta} - \frac{1}{2} P^{\alpha\beta} P^{\gamma'\delta'}\right) \left\langle \dddot{I}_{\gamma\delta}\, \dddot{I}_{\gamma'\delta'} \right\rangle d\Omega. \qquad (7)$$

Evaluate $\left(P^{\gamma}_{\alpha} P^{\delta}_{\beta} - \frac{1}{2} P_{\alpha\beta} P^{\gamma\delta}\right)\left(P^{\gamma'\alpha} P^{\delta'\beta} - \frac{1}{2} P^{\alpha\beta} P^{\gamma'\delta'}\right)$

$$= \left[P^{\gamma\gamma'} P^{\delta\delta'} - \frac{1}{2} P^{\gamma\delta} P^{\delta'\gamma'} - \frac{1}{2} P^{\gamma'\delta'} P^{\gamma\delta} + \frac{1}{4} \cdot 2 \cdot P^{\gamma\delta} P^{\gamma'\delta'} \right]$$

$$P_{\alpha\beta} P^{\alpha\beta} = (\delta_{\alpha\beta} - n_\alpha n_\beta) \times (\delta^{\alpha\beta} - n^\alpha n^\beta) = 3-1-1+1 = 2$$

$$= \left[P^{\gamma\gamma'} P^{\delta\delta'} - \frac{1}{2} P^{\gamma\delta} P^{\gamma'\delta'} \right]$$

$$\therefore \frac{dE}{dt} = -\frac{G}{8\pi c^5} \int \left(P^{\gamma\gamma'} P^{\delta\delta'} - \frac{1}{2} P^{\gamma\delta} P^{\gamma'\delta'} \right) \left\langle \dddot{I}_{\gamma\delta}\, \dddot{I}_{\gamma'\delta'} \right\rangle d\Omega.$$

$$P^{\gamma\gamma'} P^{\delta\delta'} - \frac{1}{2} P^{\gamma\delta} P^{\gamma'\delta'} = \left(\delta^{\gamma\gamma'} - n^\gamma n^{\gamma'}\right)\left(\delta^{\delta\delta'} - n^\delta n^{\delta'}\right) - \frac{1}{2}\left(\delta^{\gamma\delta} - n^\gamma n^\delta\right)\left(\delta^{\gamma'\delta'} - n^{\gamma'} n^{\delta'}\right)$$

$$\begin{aligned}
\therefore \frac{dE}{dt} = -\frac{G}{8\pi c^5} \int \Big[& \delta^{\gamma\gamma'} \delta^{\delta\delta'} \left\langle \dddot{I}_{\gamma\delta}\, \dddot{I}_{\gamma'\delta'} \right\rangle - \delta^{\gamma\gamma'} n^\delta n^{\delta'} \left\langle \dddot{I}_{\gamma\delta}\, \dddot{I}_{\gamma'\delta'} \right\rangle \\
& - \delta^{\delta\delta'} n^\gamma n^{\gamma'} \left\langle \dddot{I}_{\gamma\delta}\, \dddot{I}_{\gamma'\delta'} \right\rangle + n^\gamma n^{\gamma'} n^\delta n^{\delta'} \left\langle \dddot{I}_{\gamma\delta}\, \dddot{I}_{\gamma'\delta'} \right\rangle \\
& - \frac{1}{2} \delta^{\gamma\delta} \delta^{\gamma'\delta'} \left\langle \dddot{I}_{\gamma\delta}\, \dddot{I}_{\gamma'\delta'} \right\rangle + \frac{1}{2} \delta^{\gamma\delta} n^{\gamma'} n^{\delta'} \left\langle \dddot{I}_{\gamma\delta}\, \dddot{I}_{\gamma'\delta'} \right\rangle \\
& + \frac{1}{2} \delta^{\gamma'\delta'} n^\gamma n^\delta \left\langle \dddot{I}_{\gamma\delta}\, \dddot{I}_{\gamma'\delta'} \right\rangle - \frac{1}{2} n^\gamma n^\delta n^{\gamma'} n^{\delta'} \left\langle \dddot{I}_{\gamma\delta}\, \dddot{I}_{\gamma'\delta'} \right\rangle \Big] d\Omega
\end{aligned}$$

Use the results $\int n^{\gamma} n^{\delta} \, d\Omega = \frac{4\pi}{3} \delta^{\gamma\delta}$

$$\int n^{\alpha} n^{\beta} n^{\gamma} n^{\delta} \, d\Omega = \frac{4\pi}{15} \left[\delta^{\alpha\beta} \delta^{\gamma\delta} + \delta^{\alpha\gamma} \delta^{\beta\delta} + \delta^{\delta\alpha} \delta^{\beta\gamma} \right].$$

$$\therefore \frac{dE}{dt} = -\frac{G}{8\pi c^5} \Big[\langle \dddot{I}_{\gamma\delta} \dddot{I}^{\gamma\delta} \rangle \left[4\pi - \frac{8\pi}{3} + \frac{4\pi}{15} \right] + \langle \dddot{I}_{\gamma\delta} \dddot{I}_{\gamma'\delta'} \rangle \left[\frac{2\pi}{15} - 2\pi + \frac{4\pi}{3} \right\} \times \delta^{\gamma\delta} \delta^{\gamma'\delta'} \Big]$$

$$= -\frac{G}{8\pi c^5} \Big[\langle \dddot{I}_{\gamma\delta} \dddot{I}^{\gamma\delta} \rangle \left[\frac{24\pi}{15} \right] + \delta^{\gamma\delta} \delta^{\gamma'\delta'} \langle \dddot{I}_{\gamma\delta} \dddot{I}_{\gamma'\delta'} \rangle \frac{(-8\pi)}{15} \Big]$$

$$= -\frac{G \times 24\pi}{8\pi c^5 . 15} \Big[\langle \dddot{I}_{\gamma\delta} \dddot{I}^{\gamma\delta} \rangle - \frac{1}{3} \delta^{\gamma\delta} \delta^{\gamma'\delta'} \langle \dddot{I}_{\gamma\delta} \dddot{I}_{\gamma'\delta'} \rangle \Big]$$

$$= -\frac{G}{5c^5} \langle \dddot{\mathcal{I}}_{\gamma\delta} \dddot{\mathcal{I}}^{\gamma\delta} \rangle = -L_{GW}$$

where $\mathcal{I}_{\gamma\delta} = \left(I_{\gamma\delta} - \frac{1}{3} \delta_{\gamma\delta} I \right)$.

$$\therefore L_{GW} = \frac{G}{5c^5} \langle \dddot{\mathcal{I}}_{\gamma\delta} \dddot{\mathcal{I}}^{\gamma\delta} \rangle.$$

Equal mass binary.

$$L_{GW} = \frac{G}{5c^5}\left(\frac{MR^2}{2}\right)^2 (2\omega)^6 \left[\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right]$$

$$= \frac{G}{5c^5}\left(\frac{Ma^2}{8}\right)^2 (2\omega)^6 \cdot 2$$

$$= \frac{2GM^2a^4\omega^6}{5c^5}.$$

Elliptic orbits $\longrightarrow$ Peters-Matthews Correction factor

$f(e)$ where

$$f(e) = \frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}.$$

Rewritten formula Take $a = mn$ $m = \frac{GM}{c^2}$

n is a number.

$\omega^2 a^3 = GM$, $\omega^2 (mn)^3 = GM$, $\omega^6 = \frac{(GM)^3}{(mn)^9}$

$$L_{GW} = \frac{2}{5}\frac{M}{m}\frac{c^3}{n^5} = \frac{2}{5}\cdot\frac{c^5}{G}\cdot\frac{1}{n^5}$$

Similarly $\omega = \left(\frac{GM}{a^3}\right)^{1/2} = \frac{c}{m\,n^{3/2}}$

$$\nu = \frac{c}{2\pi m\,n^{3/2}}$$

GW frequency is $\underline{2\omega}$ or $\underline{2\nu}$

① Check values of L_{GW}, ν_{GW} etc
for GW150914, GW190521 etc.

② Work out for unequal mass binary in circular orbit (Tutorial Set 4).

Follow slides:

① Other derivations of Quadrupole formula
② Reference for Second order Riemann, Ricci Einstein and the stress energy tensor construction.